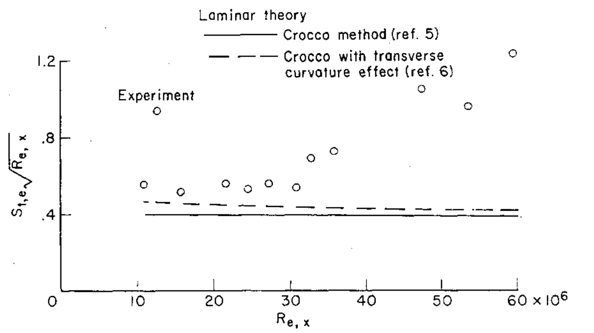

Fig. 3 Variation of heat-transfer parameter with Reynolds number at x = 58 in., $T_w/T_t \approx 0.66$.

of slight flow angularity inducing three-dimensional effects, which enhance flow instability and the onset of turbulent bursts.

Thin-film resistance gages then were installed in the same model, and heat-transfer tests were performed over a stagnation pressure range of 500 to 4000 psi. Results of this test for the heat gage at $x = 58$ in. is shown in Fig. 3. Here, a Reynolds number for transition of roughly 30×10^6 is indicated. It should be mentioned that between each of the tests described herein the model was removed from the tunnel and reinstalled. Thus, the question of alignment is pertinent. In addition, $T_w/T_t \simeq 0.66$ for the heat-transfer tests, whereas $T_w/T_t \simeq 1$ during the pitot-pressure measurements. Although all of the data are for angles of attack within $\pm 0.2°$ of zero, the range of Reynolds number for transition indicated here $[30(10)^6$ to $40(10)^6]$ is felt to be primarily a result of the extreme sensitivity of transition Reynolds number for this configuration to angle of attack rather than a result of wall temperature effects. Sensitivity of transition Reynolds number to angle of attack also has been noted previously at lower Mach numbers (see, e.g., Refs. 2–5). Although a transition Reynolds number range of $30(10)^6$ to $40(10)^6$ appears high, it should be noted from Fig. 2 that, at $R_{e,x} = 44(10)^6$, $R_{e,\theta} = 1750$ (which compares with $R_{e,\theta} = 1000$ for transition at $M = 10$ in Ref. 2).

The curves shown in Fig. 3 are theoretical estimates. The solid line is the Crocco method solution[6] for the laminar boundary layer on a flat plate modified by the Mangler transformation for flow over a cone. The dashed line is the transverse curvature correction to the Crocco solution predicted by Ref. 7 and is seen to be significant. Even with the transverse curvature correction, the theory is 15 to 20% below the experimental results, however. The extent to which the difference is a result of experimental error, entropy gradient effects from shock curvature caused by boundary-layer displacement effects near the cone tip, the three-dimensional nature of the boundary layer, or inadequacy of boundary-layer theory in the hypersonic range, is not known at present. It is hoped that future work will shed some light on this aspect of the problem.

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Numerical Treatment of Turbulent Flows

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It is in the nature of numerical models of hydrodynamic flow problems that a model should substitute finite approximations for the functions describing the hydrodynamic and thermodynamic state of the flow. A usual technique is to divide the region of flow into a finite mesh of grid points at which functional values are specified. The differential equations of hydrodynamics then are approximated by difference equations involving these grid-point values. Presently available computers permit calculations with 10^4 grid points to be carried out for 10^3 time steps in tens of hours.

It is clear that, in any finite numerical model of a turbulent hydrodynamic flow, we must be satisfied with an explicit description of only those scales of motion larger than the grid scale. All of the scales of motion smaller than grid scale must be treated, if at all, as more or less random motions superimposed on the large scale flow. Thus the appropriate division between "mean" and "eddy" flow is determined by grid size. Even though the initial state of a fluid involves only larger scales of motion, the nonlinear nature of the equations leads to motions of smaller and smaller scale. For a numerical model to be realistic it too must lead to a cascading of energy from larger to smaller scales reaching, usually quite soon, the scale of the grid. It then is necessary to rely on some statistical treatment of the further cascade process. This process can be considered as loss of energy from the larger scales of motion and thus as an energy dissipation. Perhaps the simplest solution in lieu of a greater understanding of turbulence is to introduce a viscosity to provide this dissipation. It is tempting also to consider this viscosity as an eddy viscosity describing the statistical influence of the subgrid scale motions on the explicitly described flow. Such an eddy viscosity was introduced by Smagorinsky[1] in a numerical model of the atmosphere.

The magnitude of the eddy (kinematic) viscosity coefficient ν should depend on the grid scale λ and on the specific energy cascade rate ϵ. If it depends only on these quantities, then dimensional considerations determine its form. The viscosity coefficient ν has dimensions of L^2T^{-1}, a specific energy cascade rate ϵ of L^2T^{-3}, and a scale λ of L. Letting $\nu = \alpha\epsilon^m\lambda^n$ with α dimensionless, we have $L^2T^{-1} = L^{2m}T^{-3m}L^n$, whence $m = \frac{1}{3}$, $n = \frac{4}{3}$, and $\nu = \alpha\epsilon^{1/3}\lambda^{4/3}$. We have assumed that the molecular viscosity is negligible and plays no role in determining the eddy viscosity. Should the molecular

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viscosity coefficient of the fluid be larger than this eddy viscosity coefficient, then the flow is relatively viscous already, and the problem of smaller scales of motion does not arise.

For equilibrium spectra of turbulence in a range of scales at which negligible molecular viscous dissipation is occurring, ϵ is independent of λ, and the eddy viscosity coefficient is proportional to $\lambda^{4/3}$. This relationship has abundant observational verification in the atmosphere and ocean with values of α of the order of 1.

In a numerical model we may take the scale λ to be of the order of the grid distance Δx. There remains the problem of determining ϵ. A consistent estimate of ϵ can be based on the dissipation of energy from the large scales as calculated, namely, $\epsilon = \nu D^2$. Here D^2 is a finite difference approximation to the positive scalar $2s^{ij}s_{ij}$ where

$$s_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

is the rate of strain tensor defined in terms of the derivatives of the velocity $u_{i,j}$. We are assuming here for simplicity, but somewhat arbitrarily, that the bulk coefficient of eddy viscosity is $\frac{2}{3}$ of the shear coefficient, i.e., that the second coefficient is zero. We are also assuming isotropic eddies in defining a single eddy viscosity coefficient. Both assumptions are open to question. Then

$$\nu = \alpha \nu^{1/3} D^{2/3} \lambda^{4/3}$$

or

$$\nu = \alpha^{3/2} D \lambda^2$$

The viscous stress tensor is given by

$$\sigma^{ij} = 2\nu\rho s^{ij} = 2\alpha^{3/2}\lambda^2\rho[2s^{mn}s_{mn}]^{1/2}s^{ij}$$

where, of course, finite difference approximations must be made.

For flow in one space dimension, the arguments about small eddies seem to make no sense; however, the formation of shocks has much the same formal character of a cascading of energy from larger to smaller scales of motion as a consequence of the nonlinear nature of the equations. If we specialize the preceding results to this case, then in finite difference approximation

$$s_{11} = \Delta u/\Delta x \qquad D = 2^{1/2}|\Delta u/\Delta x|$$

$$\nu = \alpha^{3/2} 2^{1/2} |\Delta u| \Delta x \qquad \sigma^{11} = (2\alpha)^{3/2} \rho |\Delta u| \Delta u$$

which leads to a viscous pressure $q = -\sigma^{11}$. This is the form of nonlinear artificial viscosity proposed in 1950 by von Neumann and Richtmyer[2] and used since in many shock hydrodynamics calculations. It would not have come from the more general formulation if the bulk coefficient of eddy viscosity had been set to zero. We see, incidentally, that the eddy viscosity coefficient has the form $\nu = l^2|u_{i,j}|$ given by Prandtl's mixing-length theory with the mixing length $l \approx \lambda \approx \Delta x$.

Finally we can recognize limitations in obtaining numerical solutions of the Navier-Stokes equations when molecular viscosity is to be large enough so that we need not introduce an eddy viscosity. If n is the number of grid points in one direction, we can estimate

$$\Delta u \approx U/n \qquad \Delta x \approx L/n$$

where U, L are characteristic velocities and lengths in the large for the flow. Then $R = UL/\nu \approx n^2$ is the largest Reynolds number for which no eddy viscosity need be introduced. Since, for three-dimensional problems, the computing time is proportional to n^4, we see that the maximum Reynolds number of proper three-dimensional Navier-Stokes computations is going to increase only as the square root of available computing speeds.

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Magnetohydrodynamic-Hypersonic Viscous and Inviscid Flow near the Stagnation Point of a Blunt Body

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IN studies of magnetohydrodynamic re-entry phenomena, the interaction between the magnetic field carried by the re-entry vehicle and the flow of the partially ionized gas surrounding the vehicle is given by the magnetic interaction parameter. This parameter is the product of the magnetic Reynolds number and the ratio of the magnetic pressures to the dynamic pressures. For a small magnetic interaction parameter the flow is essentially undisturbed by the magnetic field, and the induced magnetic field is negligible in comparison with the primary field. The magnetic field at any point in the flow can be assumed to be that of the primary field. By increasing the magnetic interaction parameter, i.e., increasing the conductivity (magnetic Reynolds number) or the applied magnetic field, the flow is no longer undisturbed. The effect, which is of concern here, is that the induced magnetic field is no longer negligible in comparison with the primary field.

An approach to simplifying the involved partial differential equations describing the hypersonic flow is to restrict the investigation to a local-similarity solution. For further computational simplification, the magnetic-field strength at the shock usually is assumed as a boundary condition. Using this approach, Bush found that numerical integration becomes impossible for the inviscid case when the value of the magnetic interaction parameter exceeds a certain critical value.[1,2] More recently, a similar result was found by Smith and Wu for the viscous case.[3]

In the present work, the magnetic field is defined at the body where it is a natural characteristic of the problem. The solution then consists of solving ordinary differential equations by a quasi-linearization technique, which simultaneously satisfies boundary conditions on the shock and on the body.

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